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ON MODULES OF BOUNDED MULTIPLICITIES FOR THE SYMPLECTIC ALGEBRAS

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ABSTRACT. Simple infinite dimensional highest weight modules having bounded weight multipicities are classified as submodules of a tensor product. Also, it is shown that a simple torsion free module of finite degree tensored with a finite dimensional module is completely reducible.

0. INTRODUCTION

Let $\mathcal{G}$ be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ with Cartan subalgebra $\mathcal{H}$. Let $\{\alpha_i \mid 1 \leq i \leq n\}$ be a basis of simple roots of $\mathcal{G}$ with respect to $\mathcal{H}$ and $\{\omega_i \mid 1 \leq i \leq n\}$ be the corresponding fundamental weights. Let $\mathcal{V}$ be a simple $\mathcal{H}$-diagonalizable $\mathcal{G}$-module having a weight space decomposition given by $\mathcal{V} = \bigoplus \sum_{\mu \in \mathrm{wt}(\mathcal{V})} \mathcal{V}_\mu$ with $\mathrm{wt}(\mathcal{V}) \subseteq \mathcal{H}^*$ denoting the set of weights of $\mathcal{V}$. The module $\mathcal{V}$ is said to have *bounded multiplicities* provided there is some $B \in \mathbb{Z}_{>0}$ such that $\dim \mathcal{V}_\mu \leq B$ for all $\mu \in \mathrm{wt}(\mathcal{V})$. In this case, the minimal B is called the *degree* of the module. Recently, in [BBL], it was shown that an algebra $\mathcal{G}$ can have a simple infinite dimensional module with bounded multiplicities if and only if it is either the Lie algebra of traceless matrices $s\ell(n, \mathbb{C})$ or the symplectic Lie algebra $sp(2n, \mathbb{C})$. Modules of degree 1 are called *completely pointed*. All simple infinite dimensional completely pointed modules are classified and explicitly constructed in [BBL].

Fernando [F] describes a construction of all simple $\mathcal{G}$-modules having a finite dimensional weight space which generalizes the usual Verma module construction for highest weight modules. Fernando's construction assumes that one knows all simple torsion free modules of finite degree. A module is *torsion free* provided the root vectors corresponding to $\mathcal{H}$ act injectively on the module. Clearly, a cyclic torsion free module having a finite dimensional weight space has bounded multiplicities and all weight spaces have dimension equal to the degree of the module.

Throughout this article, $M(\omega)$ denotes the Verma module of highest weight ω, $L(\omega)$ denotes the simple module of highest weight ω, and $\mathcal{M}$ denotes a simple completely pointed torsion free module.

Since $\mathcal{M}$ admits a central character, we know from a result of Duflo (see [D1, 7.4.8]) that this character is equal to the central character $\mathcal{X}_\omega$ of some $L(\omega)$. Examples of torsion free modules of finite degree can easily be constructed by considering the submodules of $\mathcal{M} \otimes L(\nu)$ where $L(\nu)$ is a finite dimensional module. In [K],

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Kostant studies the tensor product of an infinite dimensional module $\mathcal{V}$ admitting a central character $\mathcal{X}_\omega$ tensored with $L(\nu)$. There he shows that if $\mathrm{wt}(\nu)$ is the set of weights of $L(\nu)$, then for $\gamma \in \mathrm{wt}(\nu)$ the central characters $\mathcal{X}_{\omega+\gamma}$ of $L(\omega+\gamma)$ have the property that

$$\prod_{\gamma \in \mathrm{wt}(\nu)} (\mathcal{X}_{\omega+\gamma}(z_\gamma) - z_\gamma)(\mathcal{V} \otimes L(\nu)) = 0$$

for all z_γ chosen independently as functions of γ from the center $\mathcal{Z}$ of the universal enveloping algebra U of $\mathcal{G}$. In particular cases, this identity is still valid if the product is taken over an appropriate proper subset of $\mathrm{wt}(\nu)$.

In the case of $s\ell(n, \mathbb{C})$-modules, the tensor product $\mathcal{M} \otimes L(\nu)$ is studied in [BL1]. The $\mathcal{X}_{\omega+\gamma}$'s required to annihilate $\mathcal{M} \otimes L(\nu)$ are determined and $\mathcal{M} \otimes L(\nu)$ is shown to be completely reducible provided these central characters are distinct. Moreover, in that article, an example is presented to show that $\mathcal{M} \otimes L(\nu)$ may contain nonsimple indecomposable submodules when this condition on the central characters is not met.

In this paper, we restrict our attention to modules of $C_n = sp(2n, \mathbb{C})$. Section 1 provides general background material. In Section 2, we determine the set of all weights $\omega \in \mathcal{H}^*$ such that $L(\omega)$ has bounded multiplicities. We further show that each such $L(\omega)$ of infinite dimension is equivalent to a submodule of the tensor product module $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ for an appropriate choice of the integral dominant weight ν. In Section 3, we replace the completely pointed highest weight module $L(-\frac{1}{2}\omega_n)$ with a completely pointed torsion free module $\mathcal{M}$ and continue our study of tensor product modules of the form $\mathcal{M} \otimes L(\nu)$ with $L(\nu)$ finite dimensional. In particular, we show that $\mathcal{M} \otimes L(\nu)$ is completely reducible and that the central characters required to decompose this module are exactly the central characters occurring in the decomposition of $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$.

We conjecture that every simple torsion free module of finite degree is equivalent to a submodule of tensor product modules of the form $\mathcal{M} \otimes L(\nu)$.[1] This conjecture has been proven in [BFL] when $\mathcal{G} = s\ell(2, \mathbb{C})$.

1. Preliminaries

Let $\mathcal{G}_{2n} = gl(2n, \mathbb{C})$ be the Lie algebra of $2n \times 2n$ matrices over $\mathbb{C}$ determined by the commutator product. Let $\{E_{ij} \mid i, j = 1, 2, \ldots, 2n\}$ be the set of standard matrix units of $\mathcal{G}_{2n}$. For $X \in \mathcal{G}_{2n}$, X^t denotes its transpose. The simple Lie algebra $C_n = sp(2n, \mathbb{C})$ is the subalgebra of $\mathcal{G}_{2n}$ generated by the simple root vectors $\{X_i, X_i^t \mid X_i = E_{i,i+1} - E_{i+n+1,i+n}$ for $1 \le i \le n-1$ and $X_n = E_{n,2n}\}$. We identify C_n with this isomorphic image. Fix $\mathcal{H}$, a Cartan subalgebra having basis $\{E_{i,i} - E_{i+n,i+n} - E_{i+1,i+1} + E_{i+n+1,i+n+1} \mid 1 \le i \le n-1\} \cup \{E_{n,n} - E_{2n,2n}\}$. Let ϵ_i be the linear transformation which maps any $2n \times 2n$ diagonal matrix to its $(i,i)^{th}$ component. The set $\{\epsilon_i \mid i = 1, \ldots, n\}$ provides a basis for $\mathcal{H}^*$. A basis of simple roots of $sp(2n, \mathbb{C})$ is given by $\Delta = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n\}$ and the corresponding fundamental weights are $\{\omega_1 = \epsilon_1, \omega_2 = \epsilon_1 + \epsilon_2, \ldots, \omega_n = \epsilon_1 + \cdots + \epsilon_n\}$. Let $(*, *)$ denote the inner product on the space $\mathcal{H}^*$ determined by setting $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. We also define the map $\langle *, * \rangle : \mathcal{H}^* \times \mathcal{H}^* \longrightarrow \mathbb{C}$

[1] The C_n version of this conjecture follows from a recent preprint of Olivier Mathieu. See [M].

by setting for $\omega, \alpha \in \mathcal{H}^*$

$$\langle \omega, \alpha \rangle = 2\frac{(\omega, \alpha)}{(\alpha, \alpha)}.$$

In particular, we observe that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. For $i = 1, \ldots, n-1$, X_i is a positive root vector belonging to the simple root $\epsilon_i - \epsilon_{i+1}$ while X_n is a positive root vector belonging to the simple root $2\epsilon_n$.

Let U be the universal enveloping algebra of C_n, $\mathcal{Z}$ be its center, and U_0 be the centralizer of $\mathcal{H}$ in U.

There is a particularly useful alternate realization of C_n as a subalgebra of the Weyl algebra of rank n (see [D1]). For $1 \leq i \leq n$, let x_i be commuting variables. The Weyl algebra can be realized as the associative algebra generated by the operators ∂_i, partial differentiation with respect to x_i and multiplication operators x_i, multiplication by x_i. As a subalgebra of the Weyl algebra, the root vectors of C_n corresponding to the positive simple roots α_i are taken to be $x_{n-i}\partial_{n-i+1}$ for $i = 1, \ldots, n-1$ and $-\frac{1}{2}\partial_1^2$ for $i = n$ and those corresponding to the negative simple roots $-\alpha_i$ are $x_{n-i+1}\partial_{n-i}$ for $i = 1, \ldots, n-1$ and $\frac{1}{2}x_1^2$ for $i = n$.

Vector exponential notation such as $x^{\vec{q}}$ is used to denote the product of the x_i's raised to the corresponding coordinate powers, i.e. $x^{\vec{q}} = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$. We shall have occasion to let the q_i's be polynomials in a transcendental field extension of $\mathbb{C}$. Naturally, in this setting the partial differential operator ∂_i is carried out formally. As a first application of this realization of C_n, we provide a description of the completely pointed module $L(-\frac{1}{2}\omega_n)$. In fact let $\mathcal{V}$ denote the complex vector space spanned by all monomials $x_1^{q_1} \cdots x_n^{q_n}$ where the exponents q_i are nonnegative integers with $\sum_{i=1}^n q_i$ an even integer. If we define a C_n module structure on $\mathcal{V}$ via the above realization of C_n in terms of partial differential and multiplication operators, then it is easily seen that $\mathcal{V}$ is a simple highest weight module with highest weight 1 having a weight $-\frac{1}{2}\omega_n$. In addition we note that each monomial $x_1^{q_1} \cdots x_n^{q_n}$ is weight vector corresponding to a distinct weight.

Theorem 1.2 (See [BHL] Theorem 5.5). *Let $\nu = \sum_{i=1}^n \nu_i \omega_i$ be a dominant integral weight of C_n and $L(\nu)$ be the corresponding simple finite dimensional highest weight module. Let T_ν denote the set*

$$\{\nu - \sum_{i=1}^n d_i \epsilon_i \mid d_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n d_i \in 2\mathbb{Z}, 0 \leq d_i \leq \nu_i \text{ for } i = 1, \ldots, n-1 \text{ and } 0 \leq d_n \leq 2\nu_n + 1\}.$$

Then $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ is completely reducible with decomposition

$$L(-\frac{1}{2}\omega_n) \otimes L(\nu) = \bigoplus \sum_{\kappa \in T_\nu} L(-\frac{1}{2}\omega_n + \kappa).$$

There are two observations concerning this tensor product which we need later. The first is an application of Theorem 1.2 which gives us a refinement of Kostant's result.

Remark 1.3. According to Theorem 1.2

$$\prod_{\kappa \in T_\nu} (\mathcal{X}_{-\frac{1}{2}\omega_n + \kappa}(z_\kappa) - z_\kappa)(L(-\frac{1}{2}\omega_n) \otimes L(\nu)) = 0$$

for all $z_\kappa \in \mathcal{Z}$. Since the weights $-\frac{1}{2}\omega_n+\kappa$ are not linked, the central characters are distinct. It follows that for $\kappa, \hat{\kappa} \in T_\nu$ with $\kappa \neq \hat{\kappa}$ there exists an element $z_{\kappa,\hat{\kappa}} \in \mathcal{Z}$ such that $\mathcal{X}_{-\frac{1}{2}\omega_n+\kappa}(z_{\kappa,\hat{\kappa}}) \neq \mathcal{X}_{-\frac{1}{2}\omega_n+\hat{\kappa}}(z_{\kappa,\hat{\kappa}})$. For such a choice we see that for each $\kappa \in T_\nu$

$$L(-\frac{1}{2}\omega_n+\kappa) = \prod_{\hat{\kappa}\in T_\nu, \hat{\kappa}\neq\kappa} (\mathcal{X}_{-\frac{1}{2}\omega_n+\hat{\kappa}}(z_{\kappa,\hat{\kappa}}) - z_{\kappa,\hat{\kappa}})(L(-\frac{1}{2}\omega_n)\otimes L(\nu)).$$

We fix such a choice of $z_{\kappa,\hat{\kappa}} \in \mathcal{Z}$ and define the operator

$$\mathcal{P}_\kappa = \prod_{\hat{\kappa}\in T_\nu, \hat{\kappa}\neq\kappa} (\mathcal{X}_{-\frac{1}{2}\omega_n+\hat{\kappa}}(z_{\kappa,\hat{\kappa}}) - z_{\kappa,\hat{\kappa}})$$

which projects $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$ onto the simple submodule $L(-\frac{1}{2}\omega_n+\kappa)$.

Our next observation concerns the dimensions of weight spaces in the tensor product module $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$.

Remark 1.4. Let $v_1, \ldots, v_d$ be a basis of weight vectors of the simple finite dimensional module $L(\nu)$ with $v_1 = v^+$ having weight ν. Then for $i = 1, \ldots, d$ each vector v_i has weight $\nu - \sum_{j=1}^n \ell_j^{(i)}\alpha_j$ for certain nonnegative integers $\ell_j^{(i)}$. The vector $x^{\vec{k}} \otimes v_1$ has weight

$$\begin{aligned}\lambda_{(\vec{k}\ )} = -\frac{1}{2}\omega_n - k_n\alpha_1 - (k_{n-1}+k_n)\alpha_2 - \cdots - (k_2+\cdots+k_n)\alpha_{n-1} \\ - \frac{1}{2}(k_1+\cdots+k_n)\alpha_n + \nu.\end{aligned}$$

Let b denote the maximum of the set $\{\ell_j^{(i)} \mid i = 1, \ldots, d;\ j = 1, \ldots, n\}$. Then if $\vec{k} = (k_1, \ldots, k_n)$ with $k_i \in 2\mathbb{Z}$, $k_i \geq 2b$, and

$$\vec{s}^{(i)} = (\ell_{n-1}^{(i)} - 2\ell_n^{(i)}, \ell_{n-2}^{(i)} - \ell_{n-1}^{(i)}, \ldots, \ell_1^{(i)} - \ell_2^{(i)}, -\ell_1^{(i)}),$$

then each vector $x^{\vec{k}+\vec{s}^{(i)}} \otimes v_i$ for $i = 1, \ldots, d$ belongs in $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$ and has weight $\lambda_{(\vec{k}\ )}$. Thus, for all $\vec{k}$ with $2b \leq k_i \in 2\mathbb{Z}$ the $\lambda_{(\vec{k}\ )}$ weight space of $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$ has dimension $d = \dim L(\nu)$.

We turn now to some preliminary comments on torsion free modules. Fix any n-tuple $\vec{a} = (a_1, \ldots, a_n)$ of noninteger complex numbers and set $\mathcal{I} = \{\vec{h} = (h_1, \ldots, h_n) \mid h_i \in \mathbb{Z}, \sum_{i=1}^n h_i \in 2\mathbb{Z}\}$ then define

$$\mathcal{M}(\vec{a}) = \mathrm{span}_{\mathbb{C}}\{x^{\vec{a}+\vec{h}} = x_1^{a_1+h_1}x_2^{a_2+h_2}\cdots x_n^{a_n+h_n} \mid \vec{h} \in \mathcal{I}\}. \tag{1.5}$$

We view $\mathcal{M}(\vec{a}\ )$ as a C_n-module using the realization of C_n in the Weyl algebra. Clearly, $\mathcal{M}(\vec{a})$ is a simple completely pointed torsion free C_n-module. In fact according to [BL2], every simple completely pointed torsion free C_n-module can be constructed in this way. A motivating factor in studying $\mathcal{M}(\vec{a})\otimes L(\nu)$ is provided by the following theorem.

Theorem 1.6 (See [BL1] Theorem 1.15). *Let $\mathcal{T}$ be a degree m torsion free C_n-module.*

(i) Every submodule of $\mathcal{T}$ is torsion free.

(ii) Every quotient module of $\mathcal{T}$ is torsion free.

(iii) If $L(\nu)$ is a C_n-module of dimension d, then the tensor product module $\mathcal{T}\otimes L(\nu)$ is torsion free of degree md.

Since our infinite dimensional modules have finite dimensional weight spaces, we can reduce our discussion to a finite dimensional one by studying modules of U_0, the centralizer of $\mathcal{H}$ of the universal enveloping algebra U of C_n. The theorem which permits this follows.

Theorem 1.7 (See [L2]). *(i) If M is a simple $\mathcal{H}$-diagonalizable module, then each weight space M_λ of M is a simple U_0-module.*

(ii) If M_λ is a simple U_0-module, then there exist a unique simple $\mathcal{H}$-diagonalizable module which has a weight space isomorphic to M_λ as U_0-modules.

Evidently, part (ii) of this theorem implies that a cyclic torsion free module is simple if and only if one of its weight spaces is a simple U_0-module.

We close this section with a brief discussion on the submodules of a Verma module. For $\lambda, \mu \in \mathcal{H}^*$, we write $\mu \leq \lambda$ provided $\lambda - \mu = \sum_{i=1}^n \ell_i \alpha_i$ and each ℓ_i is a nonnegative integer. Let Φ denote the roots of C_n. For each $\sigma \in \Phi$, let R_σ denote the reflection of $\mathcal{H}^*$ in the hyperplane perpendicular to σ and $R_\sigma \cdot \lambda$ denote the affine action of σ on λ as given by $R_\sigma \cdot \lambda = R_\sigma(\lambda + \delta) - \delta$ where $\delta = \sum_{i=1}^n \omega_i$.

According to the Bernstein-Gel'fand-Gel'fand (BGG) conditions, ([D1, 7.6.23]), for $\mu \in \mathcal{H}^*$, the Verma module $M(\mu)$ is isomorphic to a submodule of the Verma module $M(\lambda)$ if and only if there exists a sequence of positive roots $\sigma_1, \ldots, \sigma_k$ such that

$$\mu = R_{\sigma_k} \circ \cdots \circ R_{\sigma_1} \cdot \lambda \leq R_{\sigma_{k-1}} \circ \cdots \circ R_{\sigma_1} \cdot \lambda \leq \cdots \leq R_{\sigma_1} \cdot \lambda \leq \lambda.$$

Finally, we state a special case of a theorem found in [J].

Theorem 1.9 (See Theorem p. 176 [J]). *Let λ and μ be weights of C_2. Then the multiplicity of $L(\mu)$ in the composition series of the Verma module $M(\lambda)$ is at most one.*

2. Highest weight modules having bounded multiplicities

In this section, we classify the simple highest weight modules of C_n having bounded multiplicities and show that each such module occurs as a submodule of $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ for an appropriate choice of the finite dimensional module $L(\nu)$. We begin by establishing a sufficient condition for a highest weight module to have bounded multiplicities.

Lemma 2.1. *If λ_i are nonnegative integers for $1 \leq i \leq n-1$, λ_n is half an odd integer and $\lambda_{n-1} + 2\lambda_n + 3 \in \mathbb{Z}_{>0}$, then the simple highest weight C_n-module $L(\lambda_1\omega_1 + \cdots + \lambda_n\omega_n)$ has bounded multiplicities and is equivalent to a direct summand of the tensor product module $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ for some choice of a dominant integral weight ν.*

Proof. Assume that $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ satisfies the conditions of the lemma. Clearly, it suffices to show that the C_n-module $L(\lambda)$ is equivalent to a direct summand of the tensor product module $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ for some choice of a dominant integral weight ν, since the multiplicities of $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ are bounded by the dimension of $L(\nu)$. By Theorem 1.2 this is equivalent to showing that there exists a dominant integral weight ν such that $\lambda = -\frac{1}{2}\omega_n + \kappa$ where $\kappa \in T_\nu$.

To this end define $\nu_1 = d_1 = 0$ or 1, $\nu_2 = d_2 = \lambda_1, \ldots, \nu_{n-1} = d_{n-1} = \lambda_{n-2}$, $d_n = \lambda_{n-1}$ and $\nu_n - d_n - \frac{1}{2} = \lambda_n$. By definition each ν_i for $i = 1, \ldots, n-1$ is a nonnegative integer. Also since λ_{n-1} is a nonnegative integer and λ_n is half integer we have $\nu_n + 1 = \frac{1}{2}(\lambda_{n-1} + \lambda_{n-1} + 2\lambda_n + 3)$ is a positive integer (i.e. ν_n is a nonnegative integer) and hence $\nu = \sum_{i=1}^{n} \nu_i\omega_i$ is a dominant integral weight. Moreover

$$\lambda = \sum_{i=1}^{n-1}(\nu_i - d_i + d_{i+1})\omega_i + (\nu_n - d_n - \frac{1}{2})\omega_n = -\frac{1}{2}\omega_n + \nu - \sum_{i=1}^{n} d_i\epsilon_i.$$

Finally by definition $d_i = \nu_i$ for $i = 1, \dots, n-1$ and since $0 < \lambda_{n-1} + 2\lambda_n + 3 = d_n + 2(\nu_n - d_n - \frac{1}{2}) + 3 = -d_n + (2\nu_n + 1) + 1$, we see that $d_n \leq 2\nu_n + 1$. Therefore, once we have selected d_1 so that the sum $\sum_{i=1}^{n} d_i$ is even, we have that $\kappa = \nu - \sum_{i=1}^{n} d_i\epsilon_i \in T_\nu$ which completes the proof. ■

The remainder of this section is devoted to showing that our sufficient condition is also a necessary one in the case of infinite dimensional modules.

Initially our focus is on C_2. Let $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = 2\epsilon_2$ so that $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\}$ is the root system of C_2 relative to $\mathcal{H}$. Fix a weight $\lambda = a\omega_1 + b\omega_2 \in \mathcal{H}^*$. In order to determine the Verma submodules of $M(\lambda)$, we need to know the affine action of the Weyl group $\mathcal{W}$ of C_2. There are four reflections

$$R_\alpha \cdot \lambda = \lambda - (a+1)\alpha; \tag{2.2}$$

$$R_\beta \cdot \lambda = \lambda - (b+1)\beta; \tag{2.3}$$

$$R_{\alpha+\beta} \cdot \lambda = \lambda - (a+2b+3)(\alpha+\beta); \tag{2.4}$$

$$R_{2\alpha+\beta} \cdot \lambda = \lambda - (a+b+2)(2\alpha+\beta) \tag{2.5}$$

and three additional nonreflection elements in $\mathcal{W}$

$$\tau_1 \cdot \lambda = \lambda - (a+2b+3)\alpha - (b+1)\beta; \tag{2.6}$$

$$\tau_2 \cdot \lambda = \lambda - (a+1)\alpha - (a+b+2)\beta; \tag{2.7}$$

$$\tau_3 \cdot \lambda = \lambda - 2(a+b+2)\alpha - (a+2b+3)\beta. \tag{2.8}$$

For C_2 Verma modules the BGG conditions can be simplified.

Lemma 2.9. *For any element $\tau \in \mathcal{W} - \{\tau_3\}$, the C_2 Verma module $M(\lambda)$ contains a submodule isomorphic to $M(\tau \cdot \lambda)$ if and only if $\tau \cdot \lambda \leq \lambda$.*

Proof. Certainly, the condition $\tau \cdot \lambda \leq \lambda$ is necessary. To see that this condition is sufficient, assume $\tau \cdot \lambda \leq \lambda$ and notice that if τ is one of the reflections in $\mathcal{W}$, then the result follows immediately from the BGG conditions. Also, if $\tau = \tau_1$, or τ_2, then we have respectively

$$\tau_1 \cdot \lambda = R_\alpha \circ R_\beta \cdot \lambda \leq R_\beta \cdot \lambda \leq \lambda,$$
$$\tau_2 \cdot \lambda = R_\beta \circ R_\alpha \cdot \lambda \leq R_\alpha \cdot \lambda \leq \lambda,$$

and once again the result follows from the BGG conditions. ■

An element of a Verma module is called a *maximal vector* if it is annihilated by all positive root vectors. The following lemma greatly simplifies the study of the submodule structure of $M(\lambda)$.

Lemma 2.10. *Every submodule of a C_2 Verma module $M(\lambda)$ is generated by its maximal vectors.*

Proof. Our proof of this result uses a modification of the proof of Lemma 1 on page 341 in [D2]. Seeking a contradiction, we assume that M is a submodule of $M(\lambda)$ which is not generated by its maximal vectors. Let N denote the submodule of M generated by the maximal vectors in M. Let $\Pi = \{\nu \in \mathcal{H}^* \mid M_\nu \neq N_\nu\}$ where M_ν (resp. N_ν) denotes the ν weight space of M (resp. N). Fix a weight $\mu_0 \in \Pi$ which is maximal with respect to the usual partial ordering on $\mathcal{H}^*$ and select a nonzero vector x in M_{μ_0} not in N_{μ_0}. Clearly x is not a maximal vector since in this case we would have $x \in N$. On the other hand, $x + N$ is a maximal vector element in M/N since μ_0 is maximal in Π. It follows that $L(\mu_0)$ occurs as a composition factor in M/N and hence also in $M(\lambda)/N$.

Since x is not a maximal vector in M there exists a sequence of positive roots $\gamma_1, \ldots, \gamma_k$ with $k \geq 1$ and corresponding root elements X_{γ_i} such that $y = X_{\gamma_k} \cdots X_{\gamma_1} x$ is a maximal vector of weight $\mu_1 = \mu_0 + \gamma_1 + \cdots + \gamma_k$. It follows then that $M(\mu_1)$ is isomorphic to a submodule of M and hence of N. Clearly $\mu_0 < \mu_1 < \lambda$ and μ_0 is linked to μ_1 since both are linked to λ. In particular, there exist distinct elements $\tau, \hat{\tau} \in \mathcal{W}$ such that

$$\tau \cdot \lambda = \mu_0 \quad \text{and} \quad \hat{\tau} \cdot \mu_1 = \mu_0.$$

If $\hat{\tau} \neq \tau_3$ then, by Lemma 2.9, $M(\mu_0)$ is a isomorphic to a submodule of $M(\mu_1)$ and hence

$$M(\lambda) \supset M \supset N \supset M(\mu_1) \supset M(\mu_0).$$

It follows that $L(\mu_0)$ occurs as a composition factor in both $M(\lambda)/N$ and N, contrary to Theorem 1.9.

If $\tau \neq \tau_3$ then, by Lemma 2.9, $M(\mu_0)$ is a isomorphic to a submodule of $M(\lambda)$ and hence

$$M(\lambda) \supset M + M(\mu_0) \supset N + M(\mu_0) \supset M(\mu_0).$$

If the x above is in $N + M(\mu_0)$ then $x = r + s$ for some $r \in N$ and $s \in M(\mu_0)$ with both r and s of weight μ_0. Since $x \notin N$, $s \neq 0$, and this implies, in particular, that s is a maximal vector in $M(\mu_0)$. Thus, $s = x - r$ is in M and hence in N which is impossible since $x = r + s$ is not in N. This means that the coset determined by x in $(M + M(\mu_0))/(N + M(\mu_0))$ is a maximal vector in this module and $L(\mu_0)$ is a composition factor in both $(M + M(\mu_0))/(N + M(\mu_0))$ and $M(\mu_0)$, contrary to Theorem 1.9.

Since one of $\hat{\tau} \neq \tau_3$ or $\tau \neq \tau_3$ holds, we have arrived at a contradiction to the existence of M. ∎

For any weight $\mu \in \mathcal{H}^*$ the Kostant partition function $K(\mu)$ is defined to be the number of ways in which μ can be expressed as a linear combination of the positive roots $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ where the coefficients are nonnegative integers. Such a linear combination is called a *partition* of μ. It is well known that the dimension of the ν weight space of the Verma module $M(\lambda)$ is $K(\lambda - \nu)$. We now list some simple properties of the Kostant partition function for C_2 which we will use to analyse the multiplicities of simple highest weight modules of C_2.

Lemma 2.11.

(a) *If $p, q, p_1, q_1 \in \mathbb{Z}_{\geq 0}$ with $p_1 \geq p$ and $q_1 \geq q$, then*

$$K(p_1\alpha + q_1\beta) \geq K(p\alpha + q\beta).$$

(b) *If $p, q \in \mathbb{Z}_{\geq 0}$ with $p \leq q$, then $K(p\alpha + q\beta) = K(p\alpha + p\beta)$.*

(c) *If $p, q \in \mathbb{Z}_{\geq 0}$ with $p \geq 2q$, then $K(p\alpha + q\beta) = K(2q\alpha + q\beta)$.*
(d) *If $k, \ell, m, p \in \mathbb{Z}_{>0}$, then*

$$\begin{aligned} N &= K((p+k+m)\alpha + (p+\ell+m)\beta) - K((p+m)\alpha + (p+\ell+m)\beta) \\ &\geq K((p+k)\alpha + (p+\ell)\beta) - K(p\alpha + (p+\ell)\beta) = N' \end{aligned}$$

Proof. Part (a) follows from the observation that any partition of $p\alpha + q\beta$ can be extended to a partition of $p_1\alpha + q_1\beta$ by adding $(p_1 - p)$ α's and $(q_1 - q)$ β's.

For (b), consider any partition of $p\alpha + q\beta$, say $p\alpha + q\beta = l(2\alpha + \beta) + m(\alpha + \beta) + n\beta + k\alpha$. It follows that $p = 2l + m + k$ and $q = l + m + n$. Therefore we have $n = q - l - m \geq q - 2l - m = q - p + k \geq q - p$. This means that every partition of $p\alpha + q\beta$ must include at least $(q - p)$ β's. We can therefore reduce any partition of $p\alpha + q\beta$ to a unique partition of $p\alpha + p\beta$ by removing $(q - p)$ β's. It follows that $K(p\alpha + q\beta) = K(p\alpha + p\beta)$.

The proof of (c) is similar to that of (b).

For part (d), notice that N is the number of partitions of $\gamma = (p+k+m)\alpha + (p+\ell+m)\beta$ which involve fewer than k α's. Also, N' is the number of partitions of $\gamma' = (p+k)\alpha + (p+\ell)\beta$ having fewer than k α's. Moreover, each partition Σ' of γ' having fewer than k α's produces a partition $\Sigma = \Sigma' + m(\alpha + \beta)$ of γ having fewer than k α's from which the inequality follows. ■

Lemma 2.12.

$$\text{(a)} \lim_{p\to\infty} \left[K((p+1)\alpha + p\beta) - K(p\alpha + p\beta)\right] = \infty,$$

$$\text{(b)} \lim_{p\to\infty} \left[K(2p\alpha + (p+1)\beta) - K(2p\alpha + p\beta)\right] = \infty.$$

Proof. For part (a), we first note that $K((p+1)\alpha + p\beta) - K(p\alpha + p\beta)$ is equal to the number of partitions of $(p+1)\alpha + p\beta$ which do not involve α. Since

$$(p+1)\alpha + p\beta = l(2\alpha + \beta) + (p + 1 - 2l)(\alpha + \beta) + (l - 1)\beta \qquad \text{for } l = 1, 2, \ldots, [[\frac{p+1}{2}]]$$

we have $K((p+1)\alpha + p\beta) - K(p\alpha + p\beta) \geq [[\frac{p+1}{2}]]$, where $[\![\ ]\!]$ denotes the greatest integer function.

Similarly for (b), we observe that $K(2p\alpha + (p+1)\beta) - K(2p\alpha + p\beta)$ is equal to the number of partitions of $2p\alpha + (p+1)\beta$ which do not involve β. Since

$$2p\alpha + (p+1)\beta = l(2\alpha + \beta) + (p + 1 - l)(\alpha + \beta) + (p - l - 1)\alpha \qquad \text{for } l = 0, 1, \ldots, p - 1$$

we have that $K(2p\alpha + (p+1)\beta) - K(2p\alpha + p\beta) \geq p$. ■

The preceding two lemmas give us a constraint on the existence of C_2-modules of bounded multiplicities.

Lemma 2.13. *If for $\lambda \in \mathcal{H}^*$ and for some $\sigma \in \Phi^+$, we have $R_\sigma \cdot \lambda < \lambda$ and $L(\lambda) \simeq M(\lambda)/M(R_\sigma \cdot \lambda)$, then the multiplicities of $L(\lambda)$ are unbounded.*

Proof. For any weight $\gamma \in \mathcal{H}^*$ with $\gamma \leq \lambda$, we have

$$\begin{aligned} \dim L(\lambda)_\gamma &= \dim M(\lambda)_\gamma - \dim M(R_\sigma \cdot \lambda)_\gamma \\ &= K(\lambda - \gamma) - K(R_\sigma \cdot \lambda - \gamma). \end{aligned}$$

If $\sigma = \alpha$ and $\gamma = \lambda - (p+1)\alpha - p\beta$, then by (2.2), $a+1 \in \mathbb{Z}_{>0}$ and for each positive integer p we have

$$\begin{aligned}\dim L(\lambda)_{\lambda-(p+1)\alpha-p\beta} &= K((p+1)\alpha + p\beta) - K((p-a)\alpha + p\beta) \\ &\geq K((p+1)\alpha + p\beta) - K(p\alpha + p\beta), \text{ by Lemma 2.11(a).}\end{aligned}$$

Applying Lemma 2.12(a), we conclude that the multiplicities of $L(\lambda)$ are unbounded.

Similarly, if $\sigma = \beta$, $\alpha + \beta$ or $2\alpha + \beta$ then we represent $\dim L(\lambda)_{\lambda-2p\alpha-(p+1)\beta}$ in terms of the Kostant partion function and apply Lemma 2.11(a) to get

$$\dim L(\lambda)_{\lambda-2p\alpha-(p+1)\beta} \geq K(2p\alpha + (p+1)\beta) - K(2p\alpha + p\beta).$$

After applying Lemma 2.12(b), we conclude once again that the multiplicities of $L(\lambda)$ are unbounded. ■

Our next result characterizes the simple highest weight C_2-modules with bounded multiplicities. It depends heavily on Lemma 2.10 and the BGG conditions. In light of the BGG conditions, equations (2.2) through (2.8) tell us that the Verma submodules of $M(\lambda)$ depend on the set

$$\mathcal{S}_\lambda = \{a+1, b+1, a+b+2, a+2b+3\} \cap \mathbb{Z}_{>0}$$

while Lemma 2.10 allows us to focus on the Verma submodules. The proof of this characterization is a case study involving $\mathcal{S}_\lambda$. There are eleven possible $\mathcal{S}_\lambda$'s: $\emptyset$, $\{a+1\}$, $\{b+1\}$, $\{a+b+2\}$, $\{a+2b+3\}$, $\{a+1, a+2b+3\}$, $\{a+1, a+b+2\}$, $\{b+1, a+2b+3\}$, $\{b+1, a+b+2, a+2b+3\}$, $\{a+1, a+b+2, a+2b+3\}$, and $\{a+1, b+1, a+b+2, a+2b+3\}$.

Theorem 2.14. *If $\lambda = a\omega_1 + b\omega_2$, then the simple C_2-module $L(\lambda)$ has bounded multiplicities if and only if either*

(a) $\mathcal{S}_\lambda = \{a+1, b+1, a+b+2, a+2b+3\}$ *in which case $L(\lambda)$ is finite dimensional, or*

(b) $\mathcal{S}_\lambda = \{a+1, a+2b+3\}$ *in which case $L(\lambda)$ is infinite dimensional and b is half an odd integer.*

Proof. We first show that the conditions on $\mathcal{S}_\lambda$ are sufficient to give us bounded multiplicities.

If $\mathcal{S}_\lambda = \{a+1, b+1, a+b+2, a+2b+3\}$, then a and b are nonnegative integers and hence $L(\lambda)$ is a finite dimensional module. Clearly then in this case $L(\lambda)$ has bounded multiplicities.

Assume now that $\mathcal{S}_\lambda = \{a+1, a+2b+3\}$. Since $a+2b+3 = \ell \in \mathbb{Z}_{>0}$, we have $b = \frac{\ell-a-3}{2}$ is a half integer. If $b \in \mathbb{Z}$ then, since $a+b+2 \notin \mathcal{S}_\lambda$, $b+1$ must be a negative integer and then $a+2b+3 < a+b+2 \leq 0$ contradicting the assumption that $a+2b+3 > 0$. It follows that b is one half of an odd integer. Therefore λ satisfies the conditions of Lemma 2.1 which asserts that $L(\lambda)$ has bounded multiplicities.

There are nine more possibilities for $\mathcal{S}_\lambda$ to be considered. If $\mathcal{S}_\lambda = \emptyset$, then equations (2.2) through (2.8), the BGG condition and Lemma 2.10 tells us that $M(\lambda)$ has no proper submodules, and so $L(\lambda) = M(\lambda)$. In this case, $\dim L(\lambda)_{\lambda-p\alpha-p\beta} = K(p\alpha + p\beta) \geq p$ and the multiplicities of $L(\lambda)$ are unbounded.

If $\mathcal{S}_\lambda$ is any of $\{a+1\}$, $\{b+1\}$, $\{a+b+2\}$, or $\{a+2b+3\}$, then $M(\lambda)$ has a unique proper submodule $M(\mu)$ where μ is equal to $R_\alpha \cdot \lambda$, $R_\beta \cdot \lambda$, $R_{2\alpha+\beta} \cdot \lambda$, $R_{\alpha+\beta} \cdot \lambda$ respectively and therefore the multiplicities of $L(\lambda)$ are unbounded by Lemma 2.13. The remaining four cases must be handled separately.

Case $\mathcal{S}_\lambda = \{a+1, a+b+2\}$. In this case $b+1$ is a negative integer and $a+2b+3$ is a nonpositive integer. If $a+2b+3=0$ then $M(\lambda)$ contains only two proper Verma submodules, namely $M(R_\alpha \cdot \lambda)$ and $M(R_{2\alpha+\beta}\cdot\lambda)$. We also have that $M(\lambda) \supset M(R_\alpha\cdot\lambda) \supset M(R_{2\alpha+\beta}\cdot\lambda)$. In this case $L(\lambda) \simeq M(\lambda)/M(R_\alpha\cdot\lambda)$ and the multiplicities of $L(\lambda)$ are unbounded by Lemma 2.13.

Therefore, we may assume that $a+2b+3<0$ and in this case $M(\lambda)$ contains three proper Verma submodules, namely $M(R_\alpha\cdot\lambda)$, $M(R_{2\alpha+\beta}\cdot\lambda)$ and $M(\tau_2\cdot\lambda)$. We also have that $M(R_\alpha\cdot\lambda)\cap M(R_{2\alpha+\beta}\cdot\lambda) = M(\tau_2\cdot\lambda)$ and hence $L(\lambda)\simeq M(\lambda)/(M(R_\alpha\cdot\lambda)+M(R_{2\alpha+\beta}\cdot\lambda))$. Therefore for any weight $\mu\in\mathcal{H}^*$ we have

$$\dim L(\lambda)_\mu = K(\lambda-\mu) - K(R_\alpha\cdot\lambda-\mu) - K(R_{2\alpha+\beta}\cdot\lambda-\mu) + K(\tau_2\cdot\lambda-\mu).$$

Since $0 < p+a+2b+3 < p < p+a+b+2$ for sufficiently large p and $b+1$ is a negative integer, Lemma 2.11(b) yields

$$\begin{aligned} K((p+a+2b+3)\alpha+p\beta) &= K((p+a+2b+3)\alpha+(p+a+2b+3)\beta)\\ &= K((p+a+2b+3)\alpha+(p+a+b+2)\beta)\end{aligned}$$

and so by Lemma 2.11(a) we have

$$\begin{aligned} &\dim L(\lambda)_{\lambda-(p+2a+2b+4)\alpha-(p+a+b+2)\beta}\\ &\quad= K((p+2a+2b+4)\alpha+(p+a+b+2)\beta) - K(p\alpha+p\beta)\\ &\qquad + K((p+a+2b+3)\alpha+p\beta) - K((p+a+2b+3)\alpha+(p+a+b+2)\beta)\\ &\quad= K((p+2a+2b+4)\alpha+(p+a+b+2)\beta) - K(p\alpha+p\beta)\\ &\quad\geq K((p+1)\alpha+p\beta) - K(p\alpha+p\beta).\end{aligned}$$

Thus by Lemma 2.12(a) we have that the multiplicities of $L(\lambda)$ are unbounded.

Case $\mathcal{S}_\lambda = \{b+1, a+2b+3\}$. In this case, we have that $a+1$ is a negative integer and $a+b+2$ is a nonpositive integer. If $a+b+2=0$, then $M(\lambda)$ contains only two proper Verma submodules $M(R_\beta\cdot\lambda)$ and $M(R_{\alpha+\beta}\cdot\lambda)$ and further $M(R_\beta\cdot\lambda)$ contains $M(R_{\alpha+\beta}\cdot\lambda)$. Thus, $L(\lambda)\simeq M(\lambda)/M(R_\beta\cdot\lambda)$ and the multiplicities of $L(\lambda)$ are unbounded by Lemma 2.13.

Therefore, we may assume that $a+b+2<0$. In this case, $M(\lambda)$ contains three proper Verma submodules, namely $M(R_\alpha\cdot\lambda)$, $M(R_{\alpha+\beta}\cdot\lambda)$ and $M(\tau_1\cdot\lambda)$. Also we have that $M(R_\beta\cdot\lambda)\cap M(R_{\alpha+\beta}\cdot\lambda) = M(\tau_1\cdot\lambda)$. Therefore

$$L(\lambda)\simeq M(\lambda)/\big(M(R_\beta\cdot\lambda)+M(R_{\alpha+\beta}\cdot\lambda)\big).$$

For any weight $\mu\in\mathcal{H}^*$ we have that

$$\dim L(\lambda)_\mu = K(\lambda-\mu) - K(R_\beta\cdot\lambda-\mu) - K(R_{\alpha+\beta}\cdot\lambda-\mu) + K(\tau_1\cdot\lambda-\mu).$$

Since $a+b+2<0$, we have that $2p+a+2b+3 > 2p > 2(p+a+b+2)\geq 0$ for sufficiently large positive integers p. In this case, we may apply Lemma 2.11(c) twice to obtain

$$\begin{aligned} &K((2p+a+2b+3)\alpha+(p+a+b+2)\beta)\\ &\qquad= K(2(p+a+b+2)\alpha+(p+a+b+2)\beta)\\ &\qquad= K(2p\alpha+(p+a+b+2)\beta).\end{aligned}$$

Thus, by Lemma 2.11(a) for all large p we have

$$\begin{aligned}
&\dim L(\lambda)_{\lambda-(2p+a+2b+3)\alpha-(p+a+2b+3)\beta} \\
&\quad = K((2p+a+2b+3)\alpha+(p+a+2b+3)\beta)-K(2p\alpha+p\beta) \\
&\qquad - K((2p+a+2b+3)\alpha+(p+a+b+2)\beta)+K(2p\alpha+(p+a+b+2)\beta) \\
&\quad = K((2p+a+2b+3)\alpha+(p+a+2b+3)\beta)-K(2p\alpha+p\beta) \\
&\quad \geq K(2p\alpha+(p+1)\beta)-K(2p\alpha+p\beta).
\end{aligned}$$

By Lemma 2.12(b), this latter term tends to infinity as p tends to infinity and hence the multiplicities of $L(\lambda)$ are unbounded.

Case $\mathcal{S}_\lambda = \{b+1, a+b+2, a+2b+3\}$. In this case, $a+1$ is a nonpositive integer. If $a+1=0$, then $M(\lambda)$ contains three proper Verma submodules

$$M(\lambda) \supset M(R_\beta\cdot\lambda) \supset M(R_{2\alpha+\beta}\cdot\lambda) \supset M(R_{\alpha+\beta}\cdot\lambda).$$

It follows that $L(\lambda) \simeq M(\lambda)/M(R_\beta\cdot\lambda)$ and the multiplicities of $L(\lambda)$ are unbounded by Lemma 2.13.

Therefore, we may assume that $a+1<0$. In this case, $M(\lambda)$ contains five proper Verma submodules: $M(R_\beta\cdot\lambda)$, $M(R_{\alpha+\beta}\cdot\lambda)$, $M(R_{2\alpha+\beta}\cdot\lambda)$, $M(\tau_1\cdot\lambda)$ and $M(\tau_3\cdot\lambda)$ with

$$M(R_\beta\cdot\lambda)\cap M(R_{2\alpha+\beta}\cdot\lambda) = M(\tau_1\cdot\lambda)+M(\tau_3\cdot\lambda)$$

and

$$M(\tau_1\cdot\lambda)\cap M(\tau_3\cdot\lambda) = M(R_{\alpha+\beta}\cdot\lambda).$$

It follows that $L(\lambda)\simeq M(\lambda)/(M(R_\beta\cdot\lambda)+M(R_{2\alpha+\beta}\cdot\lambda))$ and for any $\mu\in\mathcal{H}^*$ we have

$$\begin{aligned}
\dim L(\lambda)_\mu &= K(\lambda-\mu)-K(R_\beta\cdot\lambda-\mu)-K(R_{2\alpha+\beta}\cdot\lambda-\mu) \\
&\qquad + K(\tau_1\cdot\lambda-\mu)+K(\tau_3\cdot\lambda-\mu)-K(R_{\alpha+\beta}\cdot\lambda-\mu).
\end{aligned}$$

Since $2p > 2p+a+1 > 2(p-b-1) \geq 0$ for sufficiently large p, by Lemma 2.11(c), we have that

$$\begin{aligned}
K(2p\alpha+(p-b-1)\beta) =& K(2(p-b-1)\alpha+(p-b-1)\beta) \\
=& K((2p+a+1)\alpha+(p-b-1)\beta).
\end{aligned}$$

Similarly, since $2p+2a+2b+4 > 2p+a+1 > 2(p+a+1)\geq 0$, we have, by Lemma 2.11(c), that

$$\begin{aligned}
K((2p+a+1)\alpha+(p+a+1)\beta) =& K(2(p+a+1)\alpha+(p+a+1)\beta) \\
=& K((2p+2a+2b+4)\alpha+(p+a+1)\beta).
\end{aligned}$$

Therefore for any large positive integer p we have

$$\begin{aligned}
&\dim L(\lambda)_{\lambda-(2p+2a+2b+4)\alpha-(p+a+b+2)\beta} \\
&\quad = K((2p+2a+2b+4)\alpha+(p+a+b+2)\beta)-K(2p\alpha+p\beta) \\
&\qquad + K((2p+a+1)\alpha+(p+a+1)\beta)+K(2p\alpha+(p-b-1)\beta) \\
&\qquad - K((2p+2a+2b+4)\alpha+(p+a+1)\beta)-K((2p+a+1)\alpha+(p-b-1)\beta) \\
&\quad = K((2p+2a+2b+4)\alpha+(p+a+b+2)\beta)-K(2p\alpha+p\beta) \\
&\quad \geq K(2p\alpha+(p+1)\beta)-K(2p\alpha+p\beta) \quad \text{by Lemma 2.11(a).}
\end{aligned}$$

It follows then from Lemma 2.12 (b) that the multiplicities of $L(\lambda)$ are unbounded.

Case $\mathcal{S}_\lambda = \{a+1, a+b+2, a+2b+3\}$. In this case, we have that $b+1$ is a nonpositive integer. If $b+1=0$, then $M(\lambda)$ contains three proper Verma submodules

$$M(\lambda) \supset M(R_\alpha \cdot \lambda) \supset M(R_{\alpha+\beta} \cdot \lambda) \supset M(R_{2\alpha+\beta} \cdot \lambda),$$

$L(\lambda) \simeq M(\lambda)/M(R_\alpha \cdot \lambda)$ and the multiplicities of $L(\lambda)$ are unbounded by Lemma 2.13.

Therefore, we assume may that $b+1<0$. In this case $M(\lambda)$ contains five proper submodules: $M(R_\alpha \cdot \lambda)$, $M(R_{\alpha+\beta} \cdot \lambda)$, $M(R_{2\alpha+\beta} \cdot \lambda)$, $M(\tau_2 \cdot \lambda)$ and $M(\tau_3 \cdot \lambda)$ with

$$M(R_\alpha \cdot \lambda) \cap M(R_{\alpha+\beta} \cdot \lambda) = M(\tau_2 \cdot \lambda) + M(\tau_3 \cdot \lambda)$$

and

$$M(\tau_2 \cdot \lambda) \cap M(\tau_3 \cdot \lambda) = M(R_{2\alpha+\beta} \cdot \lambda).$$

Therefore we have that $L(\lambda) \simeq M(\lambda)/(M(R_\alpha \cdot \lambda) + M(R_{\alpha+\beta} \cdot \lambda))$ and hence for any weight $\mu \in \mathcal{H}^*$ we have

$$\begin{aligned}\dim L(\lambda)_\mu &= K(\lambda-\mu) - K(R_\alpha \cdot \lambda - \mu) - K(R_{\alpha+\beta} \cdot \lambda - \mu) \\ &\qquad + K(\tau_2 \cdot \lambda - \mu) + K(\tau_3 \cdot \lambda - \mu) - K(R_{2\alpha+\beta} \cdot \lambda - \mu).\end{aligned}$$

In particular if we take $\mu = \lambda - (p+2a+2b+4)\alpha - (p+a+b+2)\beta$, then

$$\begin{aligned}&\dim L(\lambda)_{\lambda-(p+2a+2b+4)\alpha-(p+a+b+2)\beta} \\ &\quad = K((p+2a+2b+4)\alpha + (p+a+b+2)\beta) - K((p+a+1)\alpha + (p-b-1)\beta) \\ &\qquad - K((p+a+2b+3)\alpha + (p+a+b+2)\beta) + K(p\alpha + (p-b-1)\beta) \\ &\qquad + K((p+a+2b+3)\alpha + p\beta) - K(p\alpha + p\beta).\end{aligned}$$

Now we set $\ell = -b-1$, $k = a+1$ and $m = a+2b+3$ in Lemma 2.11(d) to get

$$\begin{aligned}&K((p+2a+2b+4)\alpha + (p+a+b+2)\beta) \\ &\qquad\qquad - K((p+a+2b+3)\alpha + (p+a+b+2)\beta) \\ &\qquad\qquad \geq K((p+a+1)\alpha + (p-b-1)\beta) - K(p\alpha + (p-b-1)\beta).\end{aligned}$$

From this and Lemma 2.11(a), we have

$$\begin{aligned}&\dim L(\lambda)_{\lambda-(p+2a+2b+4)\alpha-(p+a+b+2)\beta} \\ &\qquad \geq K((p+2a+2b+4)\alpha + (p+a+b+2)\beta) - K(p\alpha + p\beta) \\ &\qquad \geq K((p+1)\alpha + p\beta) - K(p\alpha + p\beta).\end{aligned}$$

Now, apply Lemma 2.12(a) to get that the multiplicities of $L(\lambda)$ are unbounded. ■

We now use this result on C_2-modules to obtain sufficient conditions on λ for $L(\lambda)$ to be a C_n-module having bounded multiplicities.

Theorem 2.15. *If the simple highest weight module $L(\lambda_1\omega_1 + \cdots + \lambda_n\omega_n)$ of C_n is infinite dimensional and has bounded multiplicities, then λ_i is a nonnegative integer for $1 \leq i \leq n-1$, λ_n is a half an odd integer and $\lambda_{n-1} + 2\lambda_n + 3 \in \mathbb{Z}_{>0}$.*

Proof. Let $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$ and $L(\lambda)$ be an C_n-module having bounded multiplicities with maximal vector v^+. In the ϵ basis

$$\lambda_1\omega_1 + \cdots + \lambda_n\omega_n = (\lambda_1 + \cdots + \lambda_n)\epsilon_1 + (\lambda_2 + \cdots + \lambda_n)\epsilon_2 + \cdots + \lambda_n\epsilon_n.$$

The regular C_2 subalgebras determined by the simple roots $\epsilon_i - \epsilon_{i+1}$ and $2\epsilon_{i+1}$ act on v^+ to generate a highest weight C_2-module having bounded multiplicities. If

$\lambda^{(i)}$ denotes the weight of v^+ under the action of this C_2, then the coordinates of $\lambda^{(i)}$ with respect to the fundamental weights for this subalgebra are given by

$$\begin{aligned}\lambda_1^{(i)} =& \frac{2((\lambda_1\omega_1 + \cdots + \lambda_n\omega_n), \epsilon_i - \epsilon_{i+1})}{(\epsilon_i - \epsilon_{i+1}, \epsilon_i - \epsilon_{i+1})} \\ =& ((\lambda_1 + \cdots + \lambda_n)\epsilon_1 + (\lambda_2 + \cdots + \lambda_n)\epsilon_2 + \cdots + \lambda_n\epsilon_n, \epsilon_i - \epsilon_{i+1}) = \lambda_i, \text{ and} \\ \lambda_2^{(i)} =& \frac{2((\lambda_1\omega_1 + \cdots + \lambda_n\omega_n), 2\epsilon_{i+1})}{(2\epsilon_{i+1}, 2\epsilon_{i+1})} \\ =& ((\lambda_1 + \cdots + \lambda_n)\epsilon_1 + (\lambda_2 + \cdots + \lambda_n)\epsilon_2 + \cdots + \lambda_n\epsilon_n, \epsilon_{i+1}) = \lambda_{i+1} + \cdots + \lambda_n\end{aligned}$$

and hence must satisfy the conditions on a and b, respectively, in Theorem 2.14(b). It follows then that for each $1 \leq i \leq n-1$, λ_i is a nonnegative integer, $\lambda_{i+1}+\cdots+\lambda_n$ is half an odd integer and $\lambda_i + 2(\lambda_{i+1} + \cdots + \lambda_n) + 3 > 0$. This is equivalent to requiring that each of $\lambda_1, \ldots, \lambda_{n-1}$ be a nonnegative integer, λ_n be half an odd integer and $\lambda_{n-1} + 2\lambda_n + 3 > 0$. ■

3. Complete reducibility of the tensor product

In this section, we show that the tensor product of any simple completely pointed torsion free C_n-module $\mathcal{M}(\vec{a})$ with a finite dimensional module $L(\nu)$ is completely reducible.

It is convenient to work with algebras over a transcendental field extension of $\mathbb{C}$. Let $t_1, \ldots, t_n$ be algebraically independent over the complex numbers $\mathbb{C}$, $\mathbb{C}[\vec{t}\,] = \mathbb{C}[t_1, \ldots, t_n]$ be the polynomial ring in $t_1, \ldots, t_n$, and $\mathbb{C}(\vec{t}\,) = \mathbb{C}(t_1, \ldots, t_n)$ be the transcendental field extension. Let $C_n(\vec{t}\,) = C_n \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ denote the Lie algebra obtained from C_n by extension of the base field. This construction carries along with it the Cartan subalgebra $\mathcal{H}(\vec{t}\,) = \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ consisting of all diagonal matrices in $C_n(\vec{t}\,)$, the universal enveloping algebra $U(\vec{t}\,) = U \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ of $C_n(\vec{t}\,)$, the center $\mathcal{Z}(\vec{t}\,) = \mathcal{Z} \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ of $U(\vec{t}\,)$, and the centralizer $U_0(\vec{t}\,) = U_0 \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ of the Cartan subalgebra $\mathcal{H}(\vec{t}\,)$ in $U(\vec{t}\,)$. We consider C_n to be a $\mathbb{C}$-subalgebra of $C_n(\vec{t}\,)$ via the natural isomorphism $X \mapsto X \otimes 1$. This identifies U with the $\mathbb{C}$-subalgebra $U \otimes 1$ of $U(\vec{t}\,)$ by identifying u with $u \otimes 1$ for all $u \in U$. In particular, $\mathcal{Z}$ is identified with $\mathcal{Z} \otimes 1$ and $z \in \mathcal{Z}$ with $z \otimes 1 \in \mathcal{Z} \otimes 1$. $\mathcal{Z}(\vec{t}\,)$ is generated by the elements $\{c_j = c_j \otimes 1 \mid 1 \leq j \leq n\}$ where $\{c_j \mid 1 \leq j \leq n\}$ generates the center $\mathcal{Z}$ of U.

It is clear that if $\mathcal{V}$ is a C_n-module, then $\mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ is a $C_n(\vec{t}\,)$-module under the action

$$(g \otimes f_1(\vec{t}\,))(v \otimes f_2(\vec{t}\,)) = gv \otimes f_1(\vec{t}\,)f_2(\vec{t}\,)$$

where $f_1(\vec{t}\,), f_2(\vec{t}\,) \in \mathbb{C}(\vec{t}\,)$, $g \in C_n$ and $v \in V$. Moreover, the simplicity of $\mathcal{V}$ as a C_n-module implies the simplicity of $\mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\,)$ as a $C_n(\vec{t}\,)$-module.

The following notion of specialization allows us to transport information from one setting to another.

Definition 3.1. Fix an n-tuple $\vec{a} = (a_1, \ldots, a_n)$ of complex numbers. Let A be an associative algebra over the complex numbers which is generated by $\{g_\alpha \mid \alpha \in \Omega\}$ and let $M^{(\vec{a})}$ be an A-module with basis $\mathcal{B}(\vec{a}) = \{m_i^{(\vec{a})} \mid i \in I\}$. Assume $M^{(\vec{t}\,)}$ is an $A \otimes \mathbb{C}(\vec{t}\,)$-module with basis $\mathcal{B}(\vec{t}\,) = \{m_i^{(\vec{t}\,)} \mid i \in I\}$, also indexed by I. Then

$M^{(\vec{a})}$ is said to be a *specialization* of $M^{(\vec{t}\,)}$ at $\vec{t} = \vec{a}$ provided for each g_α:

$$(g_\alpha \otimes 1)m_i^{(\vec{t}\,)} = \sum_{j \in I} q_{i,j,\alpha}(\vec{t}\,)m_j^{(\vec{t}\,)} \text{ for } q_{i,j,\alpha}(\vec{t}\,) \in \mathbb{C}[\vec{t}\,], \text{ and}$$
$$g_\alpha m_i^{(\vec{a})} = \sum_{j \in I} q_{i,j,\alpha}(\vec{a})m_j^{(\vec{a})},$$

where $q_{i,j,\alpha}(\vec{a})$ is obtained from the polynomial $q_{i,j,\alpha}(\vec{t}\,)$ by substituting a_i for t_i.

Example 3.2. Fix any n-tuple $\vec{a} = (a_1, \ldots, a_n)$ of noninteger complex scalars. We recall that $\mathcal{M}(\vec{a}\,)$ denotes the completely pointed torsion free C_n-module introduced in (1.5). As usual we can consider $\mathcal{M}(\vec{a}\,)$ as a U-module. The action of U on $\mathcal{M}(\vec{a}\,)$ is determined by the action of the generating set $\{X_{\alpha_i}, Y_{\alpha_i} \mid i = 1, \ldots, n\}$ on a basis of $\mathcal{M}(\vec{a}\,)$ given by all monomials $m_{\vec{h}}^{(\vec{a}\,)} = x^{\vec{a}+\vec{h}}$ where $\vec{h} \in \mathcal{I} = \{\vec{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n \mid h_1 + \cdots + h_n \in 2\mathbb{Z}\}$.

We observe that any such module can be realized as a specialization of a $C_n(\vec{t}\,)$-module. In fact, define

$$\mathcal{M}(\vec{t}\,) = \mathrm{Span}_{\mathbb{C}(\vec{t}\,)}\{x^{\vec{t}+\vec{h}} = x_1^{t_1+h_1} \cdots x_n^{t_n+h_n} \mid \vec{h} \in \mathcal{I}\}.$$

A basis of $\mathcal{M}(\vec{t}\,)$ over $\mathbb{C}(\vec{t}\,)$ consists of all formal monomials $m_{\vec{h}}^{(\vec{t}\,)} = x^{\vec{t}+\vec{h}}$ indexed by $\vec{h} \in \mathcal{I}$. A set of generators of $U(\vec{t}\,)$ is given by $\{X_{\alpha_i} \otimes 1, Y_{\alpha_i} \otimes 1 \mid i = 1, \ldots, n\}$ and we can define a $U(\vec{t}\,)$ (or equivalently a $C_n(\vec{t}\,)$)-module structure on $\mathcal{M}(\vec{t}\,)$ by defining

$$(X_{\alpha_i} \otimes 1)x^{\vec{t}+\vec{h}} = x_{n-i}\partial_{n-i+1}x^{\vec{t}+\vec{h}} \quad \text{for } i = 1, \ldots, n-1,$$
$$(X_{\alpha_n} \otimes 1)x^{\vec{t}+\vec{h}} = -\frac{1}{2}\partial_1^2 x^{\vec{t}+\vec{h}},$$
$$(Y_{\alpha_i} \otimes 1)x^{\vec{t}+\vec{h}} = x_{n-i+1}\partial_{n-i}x^{\vec{t}+\vec{h}} \quad \text{for } i = 1, \ldots, n-1,$$
$$(Y_{\alpha_n} \otimes 1)x^{\vec{t}+\vec{h}} = \frac{1}{2}x_1^2 x^{\vec{t}+\vec{h}}.$$

Clearly then $\mathcal{M}(\vec{a}\,)$ is a specialization of $\mathcal{M}(\vec{t}\,)$ at $\vec{t} = \vec{a}$.

We now establish some general results on specializations for finitely generated algebras which require the following proposition.

Proposition 3.3. *Fix $a_1, \ldots, a_n \in \mathbb{C}$, $B \in \mathbb{Z}_{\geq 0}$ and let $p(\vec{t}\,)$ be a polynomial in $\mathbb{C}[\vec{t}\,]$. If $p(a_1+k_1, \ldots, a_n+k_n) = 0$ for all $k_i \in 2\mathbb{Z}$ with $k_i \geq B$, then $p(t_1, \ldots, t_n) = 0$.*

Proof. If $n = 1$, then the result is clear. Continuing an inductive proof, we assume that $n > 1$ and express $p(t_1, \ldots, t_n)$ as

$$p(t_1, \ldots, t_n) = \sum_j p_j(t_1, \ldots, t_{n-1})t_n^j.$$

Then when the k_i's satisfy the above conditions, we notice that the polynomial

$$p(a_1 + k_1, \ldots, a_{n-1} + k_{n-1}, t_n) = \sum_j p_j(a_1 + k_1, \ldots, a_{n-1} + k_{n-1})t_n^j$$

in one variable has infinitely many roots and so we may conclude that

$$p_j(a_1 + k_1, \ldots, a_{n-1} + k_{n-1}) = 0$$

for all k_i satisfying the given conditions. Our induction assumption tells us that each $p_j(t_1, \ldots, t_{n-1})$ is 0 and so $p(t_1, \ldots, t_n) = 0$. ■

Lemma 3.4. *Let $\mathcal{P}_\kappa$ be as defined in Remark 1.3, and select noninteger complex numbers $a_1, \ldots, a_n$. If $\mathcal{M}(\vec{a})$ is the polynomial specialization of $\mathcal{M}(\vec{t})$ at $\vec{t} = \vec{a}$, and $L(\nu)$ has dimension $d < \infty$, then*

(a) *for all choices of $\zeta_\kappa \in \mathcal{Z}(\vec{t})$ we have*

$$\prod_{\kappa \in T_\nu} \left(\mathcal{X}_{-\frac{1}{2}\omega_n + \kappa}(\zeta_\kappa) - \zeta_\kappa \right) \left(\mathcal{M}(\vec{t}) \otimes_{\mathbb{C}(\vec{t})} (L(\nu) \otimes_{\mathbb{C}} \mathbb{C}(\vec{t})) \right) = 0;$$

(b) *for all $z_\kappa \in \mathcal{Z}$ we have*

$$\prod_{\kappa \in T_\nu} \left(\mathcal{X}_{-\frac{1}{2}\omega_n + \kappa}(z_\kappa) - z_\kappa \right) (\mathcal{M}(\vec{a}) \otimes_{\mathbb{C}} L(\nu)) = 0; \textit{ and}$$

(c) $\mathcal{P}_\kappa \left(\mathcal{M}(\vec{t}) \otimes_{\mathbb{C}(\vec{t})} (L(\nu) \otimes_{\mathbb{C}} \mathbb{C}(\vec{t})) \right)$ *is a nonzero, simple torsion free $C_n(\vec{t}\,)$-module of finite degree, say d_κ.*

Proof. We assume the notation and definitions of Remarks 1.3 and 1.4. In particular $\{v_1, \ldots, v_d\}$ is a basis of weight vectors of $L(\nu)$ with v_1 being a highest weight vector of weight ν. It follows then that

$$\{x^{\vec{t}+\vec{h}} \otimes (v_i \otimes 1) \mid \vec{h} \in \mathcal{I} \text{ and } i = 1, \ldots, d\}$$

is a basis of weight vectors for the tensor product module $\mathcal{M}(\vec{t}\,) \otimes (L(\nu) \otimes \mathbb{C}(\vec{t}\,))$.

For part (a), since $\mathcal{Z}(\vec{t}\,) = \mathcal{Z} \otimes \mathbb{C}(\vec{t}\,)$, it suffices to show that for any $\vec{q} \in \mathcal{I}$ and any $i = 1, \ldots, d$ we have

$$\prod_{\kappa \in T_\nu} \left(\mathcal{X}_{-\frac{1}{2}\omega_n + \kappa}(z_\kappa) - z_\kappa \right) \left(x^{\vec{t}+\vec{q}} \otimes (v_i \otimes_{\mathbb{C}} 1) \right) = 0$$

for all arbitrarily chosen $z_\kappa \in \mathcal{Z} = \mathcal{Z} \otimes 1$. Clearly the vector on the left when expanded in terms of the basis given above can be written as

$$\sum_{\vec{h} \in \mathcal{I}} \sum_{j=1}^{d} p_{\vec{h},i,j}(\vec{t}\,) \left(x^{\vec{t}+\vec{h}} \otimes (v_j \otimes 1) \right)$$

where the coefficients $p_{\vec{h},i,j}(\vec{t}\,)$ are polynomials. Further from Remark 1.3, we know that for all $\vec{k} = (k_1, \ldots, k_n)$ where the components k_i are all sufficiently large positive even integers $p_{\vec{h},i,j}(\vec{k}\,) = 0$. Therefore, by Proposition 3.3, the $p_{\vec{h},i,j}(\vec{t}\,)$'s are 0 and (a) is established.

Part (b) follows immediately from (a) since

$$\prod_{\kappa \in T_\nu} \left(\mathcal{X}_{-\frac{1}{2}\omega_n + \kappa}(z_\kappa) - z_\kappa \right) \left(x^{\vec{a}+\vec{q}} \otimes v_i \right) = \sum_{\vec{h} \in \mathcal{I}} \sum_{j=1}^{d} p_{\vec{h},i,j}(\vec{a}\,) x^{\vec{a}+\vec{h}} \otimes v_j$$

where the polynomials $p_{\vec{h},i,j}(\vec{t}\,)$ have been shown to be 0.

In view of the statement following Theorem 1.7, for part (c), it suffices to focus on a weight space of $\mathcal{M}(\vec{t}) \otimes_{\mathbb{C}(\vec{t})} (L(\nu) \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}))$. Let $\lambda_{(\vec{t}\,)}$ denote the weight of vector $x^{\vec{t}} \otimes (v_1 \otimes 1)$. Then we see that this weight space has basis

$$x^{\vec{t}+\vec{s}^{(i)}} \otimes (v_i \otimes 1) \quad \text{for } i = 1, \ldots, d.$$

where the $\vec{s}^{(i)}$ have been defined in Remark 1.4. If $\mathcal{P}_\kappa\left(\mathcal{M}(\vec{t}\,)\otimes_{\mathbb{C}(\vec{t}\,)}(L(\nu)\otimes_{\mathbb{C}}\mathbb{C}(\vec{t}\,))\right)$ $= 0$ then for all $i = 1, \dots, d$

$$0 = \mathcal{P}_\kappa\left(x^{\vec{t}+\vec{s}^{(i)}}\otimes(v_i\otimes 1)\right) = \sum_{j=1}^{d} p_{ij}(\vec{t}\,)x^{\vec{t}+\vec{s}^{(j)}}\otimes(v_j\otimes 1)$$

which is impossible according to Remark 1.3, since we know that

$$\mathcal{P}_\kappa(x^{\vec{k}+\vec{s}^{(i)}}\otimes v_i) = \sum_{j=1}^{d} p_{ij}(\vec{k}\,)x^{\vec{t}+\vec{s}^{(j)}}\otimes v_j$$

is nonzero for certain choices of $\vec{k}$. Since the operator $\mathcal{P}_\kappa$ commutes with the action of $U(\vec{t}\,)$, $\mathcal{P}_\kappa\left(\mathcal{M}(\vec{t}\,)\otimes_{\mathbb{C}(\vec{t}\,)}(L(\nu)\otimes_{\mathbb{C}}\mathbb{C}(\vec{t}\,))\right)$ is a nonzero submodule and hence by Theorem 1.6 it is torsion free. From this it follows that all of its weight spaces are of the same dimension, say d_κ.

To check the simplicity of this module, it suffices to show that one of its weight spaces is a simple $U_0(\vec{t}\,)$-module. To this end for each $\kappa \in T_\nu$ we fix a subset $J_\kappa \subset \{1, \dots, d\}$ of cardinality d_κ such that

$$\{\mathcal{P}_\kappa\left(x^{\vec{t}+\vec{s}^{(i)}}\otimes(v_i\otimes 1)\right) \mid i \in J_\kappa\}$$

is a basis for the $\lambda_{(\vec{t}\,)}$ weight space $W^\kappa_{\lambda_{(\vec{t}\,)}}$ of $\mathcal{P}_\kappa\left(\mathcal{M}(\vec{t}\,)\otimes_{\mathbb{C}(\vec{t}\,)}(L(\nu)\otimes_{\mathbb{C}}\mathbb{C}(\vec{t}\,))\right)$. For each κ select a nonzero element in the weight space $W^\kappa_{\lambda_{(\vec{t}\,)}}$

$$0 \neq m^\kappa(\vec{t}\,) = \sum_{i\in J_\kappa} p_i^\kappa(\vec{t}\,)\mathcal{P}_\kappa\left(x^{\vec{t}+\vec{s}^{(i)}}\otimes(v_i\otimes 1)\right).$$

Without loss of generality we may assume that the coefficients $p_i^\kappa(\vec{t}\,)$ are polynomials.

We recall from Remark 1.4 that if $\vec{k} = (k_1, \dots, k_n)$ is selected so that each component k_j is an even integer greater than or equal to $2b$ for all $i = 1, \dots, d$ then the $\lambda_{(\vec{k}\,)}$ weight space of $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$ has dimension d. Let $W^\kappa_{\lambda_{(\vec{k}\,)}}$ denote the $\lambda_{(\vec{k}\,)}$ weight space of $\mathcal{P}_\kappa(L(-\frac{1}{2}\omega_n)\otimes L(\nu))$ and assume that its dimension is $\delta_\kappa(\vec{k}\,)$. For any $\vec{k}$ satisfying the conditions above we recall that $\sum_{\kappa\in T_\nu}\delta_\kappa(\vec{k}\,) = d$.

Let $\Pi(\vec{t}\,)$ denote the product of all nonzero polynomial $p_i^\kappa(\vec{t}\,)$ as both i and κ vary. By Proposition 3.3, there is a $\vec{k}$ satisfying the conditions above such that $\Pi(\vec{t}\,) \neq 0$. It follows then that for this choice of $\vec{k}$

$$0 \neq m^\kappa(\vec{k}\,) = \sum_{i=1}^{d\kappa} p_i^\kappa(\vec{k}\,)\mathcal{P}_\kappa(x^{\vec{k}+\vec{s}^{(i)}}\otimes v_i) \in W^\kappa_{\lambda_{(\vec{k}\,)}}.$$

From [L1], we know that U_0 is a finitely generated algebra. Let $G = \{g_h \mid 1 \leq h \leq s\}$ denote a generating set of U_0. Then $\{g_h\otimes 1 \mid 1 \leq h \leq s\}$ is a generating set for $U_0(\vec{t}\,)$. We further note that with our choice of $\vec{k}$ the $\lambda_{(\vec{k}\,)}$ weight space of $L(-\frac{1}{2}\omega_n)\otimes L(\nu)$ as a U_0-module is a specialization of the $\lambda_{(\vec{t}\,)}$ weight space of $\mathcal{M}(\vec{t}\,)\otimes(L(\nu)\otimes\mathbb{C}(\vec{t}\,))$ as a $U_0(\vec{t}\,)$-module at $\vec{t} = \vec{k}$.

Since $W^\kappa_{\lambda_{(\vec{k}\,)}}$ is a simple U_0-module for each $\kappa \in T_\nu$, we may select finite products π_i^κ of generators with $i = 1, \dots, \delta_\kappa(\vec{k}\,)$ such that $\{\pi_i^\kappa m^\kappa(\vec{k}\,) \mid i = 1, \dots, \delta_\kappa(\vec{k})\}$ is a basis for $W^{\kappa}_{\lambda_{(\vec{k}\ )}}$. Since the $\lambda_{(\vec{k}\ )}$ weight space of $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ is a direct sum of the $W^{\kappa}_{\lambda_{(\vec{k}\ )}}$ as κ ranges over T_ν, the d elements $\{\pi^{\kappa}_i m^{\kappa}(\vec{k}\ ) \mid \kappa \in T_\nu; i = 1, \dots, \delta_\kappa(\vec{k}\ )\}$ are linearly independent. We now expand these elements in terms of the basis $\{x^{\vec{k}+\vec{s}^{(i)}} \otimes v_i \mid i = 1, \dots, d\}$ of the $\lambda_{(\vec{k}\ )}$ weight space of $L(-\frac{1}{2}\omega_n) \otimes L(\nu)$ as follows

$$\pi^{\kappa}_i m^{\kappa}(\vec{k}\ ) = \sum_{j=1}^{d} f_{\kappa,i,j}(\vec{k}\ ) x^{\vec{k}+\vec{s}^{(j)}} \otimes v_j.$$

It follows that the $d \times d$ matrix $\left(f_{\kappa,i,j}(\vec{k}\ )\right)$ having rows labelled by κ, i and columns labelled by j has a nonzero determinant.

Now we consider the d elements $\{\pi^{\kappa}_i m^{\kappa}(\vec{t}\ ) \mid \kappa \in T_\nu; i = 1, \dots, \delta_\kappa(\vec{k}\ )\}$. If we expand these elements in terms of the basis $\{x^{\vec{t}+\vec{s}^{(i)}} \otimes (v_i \otimes 1) \mid i = 1, \dots, d\}$ then we obtain

$$\pi^{\kappa}_i m^{\kappa}(\vec{t}\ ) = \sum_{j=1}^{d} F_{\kappa,i,j}(\vec{t}\ )\ x^{\vec{k}+\vec{s}^{(j)}} \otimes (v_j \otimes 1) \in W^{\kappa}_{\lambda_{(\vec{t}\ )}}$$

where the coefficients $F_{\kappa,i,j}(\vec{t}\ )$ are polynomials with $F_{\kappa,i,j}(\vec{k}\ ) = f_{\kappa,i,j}(\vec{k}\ )$. Clearly then the $d \times d$ matrix of coefficients $F_{\kappa,i,j}(\vec{t}\ )$ has nonzero determinant. It follows that for each κ the arbitrary nonzero vector $m^{\kappa}(\vec{t}\ )$ generates $W^{\kappa}_{\lambda_{(\vec{t}\ )}}$ and hence $W^{\kappa}_{\lambda_{(\vec{t}\ )}}$ is a simple $U_0(\vec{t}\ )$-module. In addition we note that for each κ we have that $d_\kappa = \delta_\kappa(\vec{k}\ )$. ■

Theorem 3.10. *Let ν be any dominant integral weight and $\mathcal{M}(\vec{a})$ be any simple completely pointed torsion free C_n-module. Then the tensor product $\mathcal{M}(\vec{a}) \otimes L(\nu)$ is completely reducible and the decomposition into simple submodules is given by*

$$\mathcal{M}(\vec{a}) \otimes L(\nu) = \bigoplus \sum_{\kappa \in T_\nu} \mathcal{P}_\kappa \left(\mathcal{M}(\vec{a}) \otimes L(\nu)\right).$$

Proof. By Theorem 3.9 (b) we have that

$$\mathcal{M}(\vec{a}) \otimes L(\nu) = \bigoplus \sum_{\kappa \in T_\nu} \mathcal{P}_\kappa \left(\mathcal{M}(\vec{a}) \otimes L(\nu)\right)$$

and by Theorem 1.6 each nonzero submodule is torsion free. It therefore suffices to show that for each $\kappa \in T_\nu$, $\mathcal{P}_\kappa \left(\mathcal{M}(\vec{a}) \otimes L(\nu)\right)$ is nonzero and simple as a U-module or equivalently to show that some weight space of this module is a nonzero simple U_0-module. Our strategy will be to show that there is a weight space of $\mathcal{P}_\kappa \left(\mathcal{M}(\vec{a}) \otimes L(\nu)\right)$ having dimension d_κ and that the dimension of algebra of U_0 operators acting on this weight space is d_κ^2. This latter statement is equivalent to the weight space being a simple U_0-module.

We recall first that, by Lemma 3.4, for each $\kappa \in T_\nu$ the $\lambda_{(\vec{t}\ )}$ weight space $W^{\kappa}_{\lambda_{(\vec{t}\ )}}$ of $\mathcal{P}_\kappa \left(\mathcal{M}(\vec{t}) \otimes (L(\nu) \otimes_{\mathbb{C}} \mathbb{C}(\vec{t}\ ))\right)$ is a simple $U_0(\vec{t}\ )$-module. Moreover we have selected a basis for this weight space, namely

$$\mathcal{B}^{\kappa}_{\vec{t}} = \{\mathcal{P}_\kappa \left(x^{\vec{t}+\vec{s}^{(i)}} \otimes (v_i \otimes 1)\right) \mid i \in J_\kappa\}.$$

We now expand each of these basis elements in terms of the basis $\{x^{\vec{t}+\vec{s}^{(i)}} \otimes (v_i \otimes 1) \mid i = 1, \ldots, d\}$ of the $\lambda_{(\vec{t}\ )}$ weight space – i.e. for each $i \in J_\kappa$ we have

$$\mathcal{P}_\kappa \left(x^{\vec{t}+\vec{s}^{(i)}} \otimes (v_i \otimes 1) \right) = \sum_{j=1}^{d} h_{i,j}(\vec{t}\ )\ x^{\vec{t}+\vec{s}^{(j)}} \otimes (v_j \otimes 1).$$

Clearly the $d_\kappa \times d$ matrix with (i, j) component $h_{i,j}(\vec{t}\ )$ has rank d_κ. Therefore we can fix a $d_\kappa \times d_\kappa$ submatrix with determinant $D(\vec{t}\ ) \neq 0$.

Let $\rho^{\vec{t}} : U_0(\vec{t}\ ) \longrightarrow M_{d_\kappa \times d_\kappa}(\mathbb{C}(\vec{t}\ ))$ be the algebra homomorphism from $U_0(\vec{t}\ )$ into the $d_\kappa \times d_\kappa$ matrix algebra over $\mathbb{C}(\vec{t}\ )$ where, for each $\pi \in U_0(\vec{t}\ )$, $\rho^{\vec{t}}(\pi)$ is the matrix representation of the action of π on $W^\kappa_{\lambda_{(\vec{t}\ )}}$ with respect to the basis $\mathcal{B}^\kappa_{\vec{t}}$. Since for each κ, $W^\kappa_{\lambda_{(\vec{t}\ )}}$ is a simple $U_0(\vec{t}\ )$-module of dimension d_κ, there exist d_κ^2 elements $\pi^\kappa_i \in U_0$ which may be selected to be products of the generators for G such that $\rho^{\vec{t}}(\pi^\kappa_i \otimes 1)$ are $d_\kappa \times d_\kappa$ matrices which are linearly independent over $\mathbb{C}(\vec{t}\ )$. Writing the rows of the matrix $\rho^{\vec{t}}(\pi^\kappa_i)$ next to each other in ascending order, we form a $1 \times d_\kappa^2$ row vector which we call R_i. Using these row vectors, we form the $d_\kappa^2 \times d_\kappa^2$ matrix of rational functions

$$A(\vec{t}\ ) = \begin{bmatrix} R_1 \\ \vdots \\ R_{d_\kappa^2} \end{bmatrix} = \left(\frac{f_{ij}(\vec{t}\ )}{g_{ij}(\vec{t}\ )} \right).$$

Since the row vectors are linearly independent we have $\det A(\vec{t}\ ) = \frac{f(\vec{t}\ )}{g(\vec{t}\ )} \neq 0$ where $p(\vec{t}\ )$ and $q(\vec{t}\ )$ are polynomials.

Now by Proposition 3.3, we may select an n-tuple of even integers $\vec{k} = (k_1, \ldots, k_n)$ such that the polynomial

$$P(\vec{t}) = (\prod_{i,j} g_{ij}(\vec{t}\ )) f(\vec{t}\ ) g(\vec{t}\ ) D(\vec{t}\ )$$

does not vanish when $\vec{a} + \vec{k}$ is substituted for $\vec{t}$, i.e. $P(\vec{a} + \vec{k}) \neq 0$.

In particular, since $D(\vec{a} + \vec{k}\ ) \neq 0$,

$$\mathcal{B}(\vec{a} + \vec{k}\ ) = \{\mathcal{P}_\kappa(x^{\vec{a}+\vec{k}+\vec{s}^{(i)}} \otimes v_i) \mid i \in J_\kappa\}$$

is linearly independent and hence forms a basis of the $\lambda_{(\vec{a}+\vec{k}\ )}$ weight space $W^\kappa_{\lambda_{(\vec{a}+\vec{k}\ )}}$ of $\mathcal{P}_\kappa(\mathcal{M}(\vec{a}) \otimes L(\nu))$ containing $x^{\vec{a}+\vec{k}} \otimes v_1$. It follows that $W^\kappa_{\lambda_{(\vec{a}+\vec{k}\ )}}$ is a specialization of $W^\kappa_{\lambda_{(\vec{t}\ )}}$ at $\vec{t} = \vec{a} + \vec{k}$. Therefore for $i = 1, \ldots, d_{\kappa^2}$ we have that

$$\pi^\kappa_i \mathcal{P}_\kappa(x^{\vec{a}+\vec{k}+\vec{s}^{(j)}} \otimes v_j) = \sum_{k \in J_\kappa} \frac{p_{ijk}(\vec{a} + \vec{k}\ )}{q_{ijk}(\vec{a} + \vec{k}\ )}\ \mathcal{P}_\kappa(x^{\vec{a}+\vec{k}+\vec{s}^{(k)}} \otimes v_k)$$

with $p_{ijk}(\vec{a} + \vec{k}\ ), q_{ijk}(\vec{a} + \vec{k}\ ) \in \mathbb{C}$ and $g_{ijk}(\vec{a} + \vec{k}\ ) \neq 0$. Now we define the algebra homomorphism $\rho^{\vec{a}+\vec{k}} : U_0 \longrightarrow M_{d_\kappa \times d_\kappa}(\mathbb{C})$ where $\rho^{\vec{a}+\vec{k}}(\pi^\kappa_i)$ denotes the $d_\kappa \times d_\kappa$ matrix of the operator π^κ_i acting on $W^\kappa_{\lambda_{(\vec{a}+\vec{k}\ )}}$ with respect to the basis $\mathcal{B}(\vec{a} + \vec{k})$. Finally, we construct the $d_\kappa^2 \times d_\kappa^2$ matrix $A(\vec{a} + \vec{k}\ )$ in a manner similar to that used to construct $A(\vec{t}\ )$. It follows that $\det A(\vec{a} + \vec{k}\ ) = \frac{f(\vec{a}+\vec{k}\ )}{g(\vec{a}+\vec{k}\ )} \neq 0$ and hence $\{\rho^{(\vec{a}+\vec{k}\ )}(\pi^\kappa_i) \mid i = 1, \ldots, d_\kappa^2\}$ are linearly independent. This implies that the

$\lambda_{(\vec{a}+\vec{k}\ )}$ weight space $W^{\kappa}_{\lambda_{(\vec{a}+\vec{k}\ )}}$ is a simple U_0-module and hence $\mathcal{P}_\kappa\left(\mathcal{M}(\vec{a}) \otimes L(\nu)\right)$ is a simple U-module as claimed. ■

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